BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 13.5

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CONTENTS

I. (IM)PRIMITIVE PERMUTATION GROUPS

Recall that a permutation group G is a subgroup of S_d for some d, or more generally a subgroup of $\mathrm{Sym}(X)$ for some set X. The permutation group G is transitive if its action on $\{1, \dots, d\}$ is transitive, i.e., given $i, j \in \{1, \dots, d\}$, there exists some $\sigma \in G$ such that $\sigma(i) = j$.

Definition 1.

(1) Let G be a transitive permutation group on a finite set A. A block of G is a nonempty subset B of A such that for all $\sigma \in G$, either $\sigma(B) = B$ or $\sigma(B) \cap B = \varnothing$, where $\sigma(B) = \{\sigma(b) : b \in B\}$.

(2) G is primitive if the only blocks of G are are the trivial ones: the sets of size 1, and A itself. A permutation group that is not primitive is said to be imprimitive.

Remark 2. The idea is that if G is imprimitive, then we can partition $1, 2, \dots, d$, and G acts the same on any labels that are in the same part. In this way, we can understand the action of G on $1, \dots, d$ through its action on the parts, which are fewer in number. In this way, we have in some sense reduced the problem to a smaller permutation group.

Example 3. Let $A = \{1, 2, 3, 4\}$. You'll show for homework that S_4 is primitive, but D_8 is not, where we consider D_8 as a subgroup of S_4 via its action on the four vertices of a square. [Do D_8 example in Magma, showing minimal partition.]

Lemma 4. *Let G be a transitive permutation group on a finite set A.*

(1) If B is a block and $a \in B$, then the set

$$\mathrm{Stab}_G(B) = \{\sigma \in G \mid \sigma(B) = B\}$$

is a subgroup of G containing $\mathrm{Stab}_G(a)$.

(2) If B is a block and $\sigma_1(B), \sigma_2(B), \dots, \sigma_n(b)$ are the distinct images of B under elements of G, then these form a partition of A.

(3) G is primitive if and only if for each $a \in A$, the only subgroups of G containing $\mathrm{Stab}_G(a)$ are $\mathrm{Stab}_G(a)$ and G itself, i.e., $\mathrm{Stab}_G(a)$ is a maximal *subgroup of G.*

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Proof. Homework. □

II. (IM)PRIMITIVE BELYI MAPS AND PERMUTATION TRIPLES

Given a Belyi map $\varphi : X \to \mathbb{P}^1$ of degree d, let $\sigma = (\sigma_0, \sigma_1, \sigma_\infty)$ be the corresponding permutation triple. In more detail: the map φ induces a monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to S_d$ via path lifting, and σ_0, σ_1, and σ_∞ are the images of small loops around $0, 1, \infty$, respectively, under ρ. Let

$$G = \langle \sigma \rangle = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle \leq S_d$$

be the monodromy group of φ, i.e., the image of ρ.

If G is imprimitive, we want to find its "primitivization," the associated permutation group on a minimal partition of $\{1, \ldots, d\}$ by blocks. This will produce new permutation triple σ', corresponding to a new Belyi map. Here is the path to victory for this project.

(1) For each Belyi map in the LMFDB, determine if its monodromy group is primitive or not.
(2) If it isn't primitive, compute its primitivization and find the associated primitive Belyi map in the database.
(3) Record which maps arise as lifts of which primitive Belyi maps.
(4) Compare the equation of a Belyi map to the equation of its primitivization. Do we see any patterns?